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Bifurcations under Nondegenerated Conditions of Higher Degree and a New Simple Proof of the Hopf–Neimark–Sacker Bifurcation Theorem

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This paper generalizes the nondegenerated conditions that imply the most common bifurcations in uniparametric families of maps defined on $\mathbb{R}$. It also presents a new very simple proof of the Hopf bifurcation theorem of maps on $\mathbb{R}^2$ (see G. Iooss, "Mathematical Studies," Vol. 36, (1979)), based on one of the results obtained in one dimension and generalizes one of the nondegenerated conditions (the Hopf condition) of the theorem.

1. INTRODUCTION

Let us consider uniparametric families of maps

$$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

or

$$f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,$$

where $\mu \in \mathbb{R}$ will be the parameter and $x \in \mathbb{R}$ or $\mathbb{R}^2$ will be the variable in the state space under consideration.

In most cases, the function f whose dynamics we are studying could be modeling an experimental phenomenon. But f could only be an approximation of the real model g. Therefore, we are interested in knowing if functions, which are "near" f, in some sense, present similar dynamics.

Then we are interested in the change in the dynamics of the maps of the family when the parameter is varied.

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Roughly speaking, when there is a qualitative change at $\mu_0 \in \mathbb{R}$, one says that μ_0 is a *bifurcation value* or that a *bifurcation* occurs at μ_0. More rigorously, we can state the following two definitions:

DEFINITION 1. Two maps f and g in $\mathrm{Diff}(\mathbb{R}^m)$ are *topologically equivalent* or *conjugate* if there is a homeomorphism $h: \mathbb{R}^m \to \mathbb{R}^m$ with $h \circ f = g \circ h$.

And as a consequence we can state

DEFINITION 2. The appearance of a topologically nonequivalent system under variation of the parameter is called a *bifurcation*.

We will call a partition of the state space into orbits a *phase portrait* of the family. It is desirable to obtain a graphic that reflects this change in the topological type of the systems. This picture is called *the bifurcation diagram*.

DEFINITION 3. A *bifurcation diagram* of the family is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each system.

In general, a fixed point $p \in \mathbb{R}^m$ of f in $\mathrm{Diff}(\mathbb{R}^m)$ is *hyperbolic* if the linearization $Df(p)$ has no eigenvalues of unit modulus. From the theorem of Hartman–Grobman it follows (see [4] or [9]) that to study local bifurcations of fixed points in parametric families $f_\mu(x)$ it suffices to consider those parameters μ_0 for which the corresponding map has a nonhyperbolic fixed point p_0. The simplest ways in which a fixed point of a map can be nonhyperbolic are the following:

1. There is an eigenvalue equal to 1, and the remaining $m - 1$ eigenvalues have modulus not equal to 1.

2. There is an eigenvalue equal to -1, and the remaining $m - 1$ eigenvalues have modulus not equal to 1.

3. There are two complex conjugate eigenvalues having modulus 1 (which are not one of the first four roots of the unity), and the remaining $m - 2$ eigenvalues have modulus not equal to 1.

Without loss of generality, we may assume that $\mu_0 = 0$ and $p_0 = 0$. Therefore, we will give all our theorems under that assumption.

In a uniparametric family of the form

$$f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

there are two types of local bifurcations that can occur, corresponding to the eigenvalue 1 or -1. These two types will be treated in Sections 1 and 2, respectively.

On the other hand, in a uniparametric family of the form

$$f: \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2,$$

we will analyze the case of the appearance of a pair of eigenvalues of modulus 1. The phenomenon we will treat in Section 3 is called the Hopf bifurcation for maps of $\mathbb{R}^2$ or the Neimark–Sacker bifurcation. We will call this the Hopf–Neimark–Sacker bifurcation.

We will call *nondegenerated conditions* those statements that are sufficient for the appearance of a bifurcation in a family with a nonhyperbolic fixed point. The purpose of this paper is to give some new nondegenerated conditions that generalize the classical ones. As a consequence, we will be able to recognize if certain families present some of the generic bifurcations (here, "generic" means the bifurcations under consideration) in more different ways.

These generalizations have been obtained easily from the proofs given in [13] or [14]. It would be more difficult to obtain them using the normal forms theory (introduced by Arnold), as one can see in [1], where the appearance of a flip bifurcation under nondegenerated conditions of fifth order is proved, and the corresponding normal form is found.

The main interest of this paper is to give a new proof of the classical Hopf–Neimark–Sacker bifurcation theorem, where the generalized pitchfork bifurcation theorem is used to demonstrate this result in a simple way.

Of course, all the results we describe for bifurcations of fixed points can be applied to periodic points by considering the relevant iterate of the map. By using center manifold theory (see [8], [2], or [10]), they can also be seen to describe the bifurcations of maps of $\mathbb{R}^m$ or, more generally, of a Banach space.

The analogous cases of bifurcation in continuous dynamical systems can be generalized, in the same way, following a similar sketch of our proofs.

2. AN EIGENVALUE EQUAL TO 1

In this section we analyze three phenomena that can occur when a nonhyperbolic fixed point with an eigenvalue equal to 1 exists. We will call them *fold*, *transcritical*, and *pitchfork bifurcation*.

The common characteristic will be the change in the number of fixed points when we cross a certain parameter value, the bifurcation value.

Fold Bifurcation

We describe with the following definition the change that this bifurcation implies:

DEFINITION 4. A uniparametric family $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ undergoes a fold bifurcation if the family possesses a unique curve of fixed points in the $x-\mu$ plane passing through the bifurcation point that locally lies on one side of $\mu = 0$.

The following result gives the general conditions required for a family to present a fold bifurcation.

THEOREM 1. *Suppose that a one-parameter family* $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of* C^r *maps, with* $r = \max\{2n, 2m+1\}$, *has at* $\mu_0 = 0$ *the fixed point* $x_0 = 0$ *and let* $f_x(0,0) = 1$.

Assume that the following nondegenerated conditions are satisfied:

$$(\mathrm{F}_1) \quad f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0,$$
$$f_{x^{2n}}(0,0) \neq 0$$
$$(\mathrm{F}_2) \quad f_{\mu}(0,0) = f_{\mu\mu}(0,0) = \cdots = f_{\mu^{2m}}(0,0) = 0,$$
$$f_{\mu^{2m+1}}(0,0) \neq 0.$$

Then the family undergoes a fold bifurcation.

Proof. Because of the differentiability of the function f and the second nondegenerated condition, we can write the following Taylor expansion series:

$$f(0,\mu) = \frac{1}{(2m+1)!} f_{\mu^{2m+1}}(0,0)\mu^{2m+1} + O(\mu^{2m+1}).$$

This function is injective "near" $\mu = 0$ and, thanks to the first term of this series, has an odd power.

Then we can make the change of parameter given by

$$\nu = \nu(\mu) = \frac{1}{(2m+1)!} f_{\mu^{2m+1}}(0,0)\mu^{2m+1} + O(\mu^{2m+2})$$

and consider the inverse function $\mu(\nu)$. So, we will obtain a new one-parameter family, equivalent to our initial family

$$\tilde{f}(x,\nu) = f(x,\mu(\nu)),$$

which satisfies $\tilde{f}_\nu(0,0) = 1 \neq 0$ and all of the conditions with respect to x that are verified by f.

Therefore, we can suppose, to abbreviate the proof, that $f_{x^{2n}}(0,0) > 0$ and $f_\mu(0,0) > 0$. However, we will see how the signs influence the bifurcation diagrams.

Consider the function

$$g(x, \mu) = f(x, \mu) - x.$$

Thanks to the properties of f we have

(a) $g(0,0) = 0$.

(b) $g_x(0,0) = 0$.

(c) $g_{xx}(0,0) = \cdots = g_{x^{2n-1}}(0,0) = 0$, $g_{x^{2n}}(0,0) > 0$.

(d) $g_\mu(0,0) > 0$.

Because of (a) and (d), the Implicit Function Theorem gives a unique C^{2n} curve $\mu(x)$ for x near zero with

$$\mu(0) = 0$$
$$g(x, \mu(x)) = 0.$$

Differentiating that equation and evaluating each equation in $x = 0$ successively, we will find

$$\frac{d\mu}{dx}(0) = \frac{d^2\mu}{dx^2}(0) = \cdots = \frac{d^{2n-1}}{dx^{2n-1}}(0) = 0, \qquad \frac{d^{2n}\mu}{dx^{2n}}(0) < 0.$$

So $\mu(x)$ has a maximum at $x = 0$ and the existence statements follow.

For the stability of the fixed points, we consider the equation that defines these points:

$$f(x, \mu(x)) = x.$$

Now, differentiating this we obtain

$$f_x + f_\mu \frac{d\mu}{dx} = 1,$$

or in other words

$$f_x = 1 - f_\mu \frac{d\mu}{dx}.$$

Clearly, for $x > 0$, $d\mu/dx < 0$, and for $x < 0$, $d\mu/dx > 0$.

The two inequalities, together with, under our supposition, $f_\mu > 0$, lead to the conclusion that the positive fixed points are unstable and the negative fixed points are stable. ∎

Remark 1. For $m = 0$ and $n = 1$, a sketch of the proof can be seen in [13], [3], or [14].

Remark 2. If we change the signs of the nondegenerated conditions, which we have supposed, we will have the following different cases:

1. With a reversal of one of these two inequalities, the side of $\mu = 0$, where the curve of fixed points lies, reverses. In particular, if we reverse the inequality corresponding to the derivative with respect to x, then the stability of the fixed points also reverses.
2. With a reversal of both inequalities, only the stability of the fixed points reverses.

Therefore, we have four different bifurcation diagrams.

Transcritical Bifurcation

In many applications the class of maps to be considered is likely to be restricted in some way and bifurcations different from that above may occur. Perhaps the most common restriction is that the origin should be fixed for all values of the parameter, and in this case we have a *transcritical bifurcation*, as the following theorems will describe.

As above, we explain by the next definition the changes that imply this bifurcation:

DEFINITION 5. A uniparametric family $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ undergoes a transcritical bifurcation if the family possesses two curves of fixed points in the $x-\mu$ plane passing through the origin and existing on both sides of $\mu = 0$.

The following result provides the general conditions required for a family to have a transcritical bifurcation.

THEOREM 2. *Suppose that a one-parameter family* $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of* C^r *maps, with* $r = \max\{2n, 2m + 2\}$, *has the fixed point* $x_0 = 0$ *for all* μ *and let* $f_x(0,0) = 1$.

Assume that the following nondegenerated conditions are satisfied:

$$(\mathrm{T}_1) \quad f_{xx}(0,0) = f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0, \quad f_{x^{2n}}(0,0) \neq 0$$

$$(\mathrm{T}_2) \quad f_{x\mu}(0,0) = f_{x\mu\mu}(0,0) = \cdots = f_{x\mu^{2m}}(0,0) = 0, \quad f_{x\mu^{2m+1}}(0,0) \neq 0.$$

Then the family undergoes a transcritical bifurcation.

Proof. Using techniques similar to those in the proof of the fold bifurcation theorem, one can deduce this one. ∎

Remark 3. When $m = 0$ and $n = 1$, we have the known result given in [14].

Remark 4. Again, if we change the signs of the nondegenerated conditions, which we have supposed, we will arrive at the following different cases:

1. With a reversal of one of these two inequalities, the slope of the curve $\mu(x)$ reverses. In particular, if we reverse the inequality corresponding to the derivative with respect to x, then the stability of the fixed points also reverses.

2. With a reversal of both inequalities, only the stability of the fixed points reverses.

Therefore, we have four bifurcation diagrams.

Pitchfork Bifurcation

A second possibility is that the map should be restricted by a symmetry such as $f(-x, \mu) = -f(x, \mu)$; that is, f is an odd function of x. In this case we necessarily have a trivial fixed point, $f(0, \mu) = 0$, but the previous result does not apply since we now have $f_{xx}(0,0) = 0$. Instead we will have a *pitchfork bifurcation*.

DEFINITION 6. A uniparametric family $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ undergoes a pitchfork bifurcation if the family possesses two curves of fixed points in the x–μ plane passing through the bifurcation point; one curve exists on both sides of $\mu = 0$, and the other lies locally on one side of $\mu = 0$.

THEOREM 3. *Suppose that a one-parameter family* $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of* C^r *maps, with* $r = \max\{2n+1, 2m+2\}$, *is an odd function of* x, *and let* $f_x(0,0) = 1$.

Assume that the following nondegenerated conditions are satisfied:

$$(\mathrm{P}_1)\quad f_{xxx}(0,0) = \cdots = f_{x^{2n-1}}(0,0) = 0, \qquad f_{x^{2n+1}}(0,0) \neq 0$$

$$(\mathrm{P}_2)\quad f_{x\mu}(0,0) = f_{x\mu\mu}(0,0) = \cdots = f_{x\mu^{2m}}(0,0) = 0,$$

$$f_{x\mu^{2m+1}}(0,0) \neq 0.$$

Then the family undergoes a pitchfork bifurcation.

Proof. A demonstration similar to the case of the transcritical bifurcation leads us to the result (see [11]). ∎

Remark 5. As in the two preceding cases, a sketch of the proof of the case $m = 0$ and $n = 1$ can be seen in [14].

Remark 6. As in the previous cases, if we change the signs of the nondegenerated conditions, which we have supposed, we will obtain four different bifurcation diagrams.

3. AN EIGENVALUE EQUAL TO -1

The type of bifurcation we will treat in this section is called *flip* or *period-doubling bifurcation*. In this type the main characteristic is not the change in the number of fixed points. Now the main phenomenon will be the appearance of one orbit of period two.

We describe this change with the next definition.

DEFINITION 7. A uniparametric family $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ undergoes a flip bifurcation if the family possesses two curves in the $x-\mu$ plane passing through the bifurcation point; one curve of fixed points exists on both sides of $\mu = 0$ and the other one, of periodic points of period two, lie locally on one side of $\mu = 0$.

One more we have obtained the general conditions required for a family to undergo a flip or period-doubling bifurcation.

THEOREM 4. *Suppose that a one-parameter family* $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of* C^r *maps, with* $r = \max\{2n + 1, 2m + 2\}$, *has at* $\mu_0 = 0$ *the fixed point* $x_0 = 0$, *and let* $f_x(0,0) = -1$.

Assume that the following nondegenerated conditions are satisfied:

$$(\mathrm{PD}_1) \quad (f^2)_{xxx}(0,0) = (f^2)_{x^4}(0,0) = \cdots = (f^2)_{x^{2n}}(0,0) = 0, \quad (f^2)_{x^{2n+1}}(0,0) \neq 0$$

$$(\mathrm{PD}_2) \quad f_{x\mu}(0,0) = f_{x\mu\mu}(0,0) = \cdots = f_{x\mu^{2m}}(0,0) = 0, \quad f_{x\mu^{2m+1}}(0,0) \neq 0.$$

Then the family undergoes a flip or period-doubling bifurcation.

Proof. Techniques similar to those used in the other cases lead to the proof. ∎

Remark 7. For $m = 0$ and $n = 1$, a sketch of the proof can be seen in [3], [13], or [14].

Remark 8. As in the previous cases, if we change the signs of the nondegenerated conditions, which we have supposed, we will have four different bifurcation diagrams.

4. A PAIR OF EIGENVALUES OF MODULUS 1: THE HOPF–NEIMARK–SACKER BIFURCATION

To explain this type we demonstrate a more general result, which used to be called the *nonstandard Hopf bifurcation*. However, we will state the theorem of the Hopf bifurcation, because this phenomenon appears in very important models like the delayed logistic map (see [5]). It is also of interest because it describes, via a Poincaré map, the bifurcation of a limit cycle in a vector field into an invariant torus and the resulting dynamics on the torus (see [6]).

The following theorem shows that under certain conditions, an invariant circle bifurcates from the origin.

THEOREM 5. *Suppose that a one-parameter family of two-dimensional* C^s *maps, with* $s = \max\{2p+4, 2n+2\}$, $f\colon \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, *has the fixed point* $x_0 = (0,0)$ *for all* μ *sufficiently near zero and that* $D_x f(0,\mu)$ *has two nonreal eigenvalues* $\lambda(\mu)$ *and* $\overline{\lambda(\mu)}$ *such that, at* $\mu = 0$, *we have* $|\lambda(0)| = |\overline{\lambda(0)}| = 1$.

Assume that the following nondegenerated conditions are satisfied:

1. $\lambda = \lambda(0)$ *is not an mth root of unity for* $m = 1, 2, 3, \ldots, (2p+2)$ (*nonresonance condition*).

2. $(d|\lambda|/d\mu)(0) = (d^2|\lambda|/d\mu^2)(0) = \cdots = (d^{2n}|\lambda|/d\mu^{2n})(0) = 0$, $(d^{2n+1}|\lambda|/d\mu^{2n+1})(0) \neq 0$ (*new Hopf condition*).

Then there is a smooth μ*-dependent change in coordinates, bringing f into the form*

$$f(x,\mu) \sim g(z,\bar{z},\mu) = \lambda(\mu)z + \sum_{m=1}^{p} c_{2m+1}(\mu) z^{m+1}\bar{z}^m + O(|z|^{2p+1}),$$

$$z \in \mathbb{C},\ \mu \in \mathbb{R},$$

and there are $a_3, a_5, \ldots, a_{2p+1}, b_2, b_4, \ldots, b_{2p}, \phi_0, \phi_1$ *such that, in polar coordinates,*

$$g(r,\theta;\mu) = \Bigg((1+\mu)r - \sum_{m=1}^{p} a_{2m+1} r^{2m+1} + O(\mu r^3 + r^{2p+2}),$$

$$\theta + \phi_0 + \phi_1\mu + \sum_{m=1}^{p} b_{2m} r^{2m} + O(\mu^2 + \mu r^2 + r^{2p+1})\Bigg).$$

Moreover, if some of the coefficients a_{2m+1} *are not zero and* a_{2q+1} *is the first of them, then*

- $a_{2q+1} > 0 \Rightarrow f(x, \mu)$ *presents an attracting invariant circle for all sufficiently small positive* μ.
- $a_{2q+1} < 0 \Rightarrow f(x, \mu)$ *presents a repelling invariant circle for all sufficiently small negative* μ.

The bifurcation is said to be *supercritical* if the circle exists for $\mu > 0$ and *subcritical* if it exists for $\mu < 0$. Figure 1 illustrates both cases.

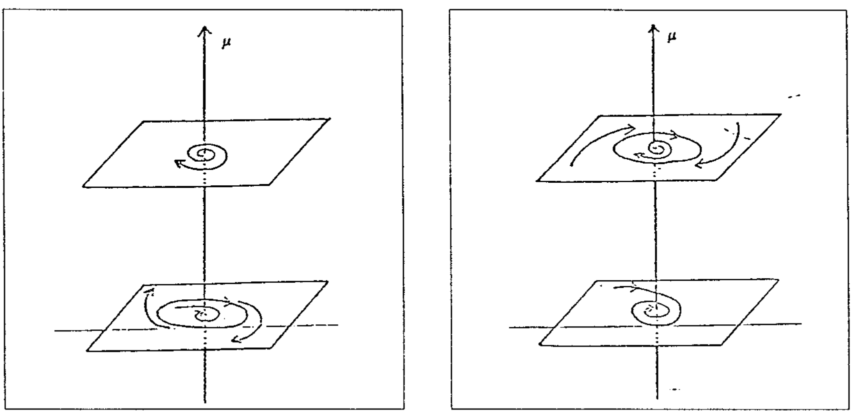
Proof. It is not very difficult (with the help of [12], [6], [14], [7], or especially [11], where a simplification of the long calculation appears) to arrive at the expression in polar coordinates given before. We will explain this briefly.

Under our assumptions, we can find a linear transformation that puts $D_x f(0, 0, \mu)$ in the form

$$Df(0,0,\mu) \sim \begin{pmatrix} \operatorname{Re} \lambda(\mu) & -\operatorname{Im} \lambda(\mu) \\ \operatorname{Im} \lambda(\mu) & \operatorname{Re} \lambda(\mu) \end{pmatrix}.$$

Letting

$$\begin{aligned} \operatorname{Re} \lambda(\mu) &= |\lambda(\mu)| \cos 2\pi\phi(\mu) \\ \operatorname{Im} \lambda(\mu) &= |\lambda(\mu)| \sin 2\pi\phi(\mu), \end{aligned}$$



Subcritical case $(a_{2q+1} < 0)$ **Supercritical case** $(a_{2q+1} > 0)$

FIG. 1. (Left) Subcritical case ($a_{2q+1} < 0$). (Right) Supercritical case ($a_{2q+1} > 0$).

it is easy to see that

$$Df(0,0,\mu) \sim |\lambda(\mu)| \begin{pmatrix} \cos 2\pi\phi(\mu) & -\sin 2\pi\phi(\mu) \\ \sin 2\pi\phi(\mu) & \cos 2\pi\phi(\mu) \end{pmatrix}.$$

As $f(0,0,\mu) = (0,0)$, we can put

$$f(x,y,\mu) = |\lambda(\mu)| \begin{pmatrix} \cos 2\pi\phi(\mu) & -\sin 2\pi\phi(\mu) \\ \sin 2\pi\phi(\mu) & \cos 2\pi\phi(\mu) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x,y,\mu) \\ f_2(x,y,\mu) \end{pmatrix},$$

where the f_i are nonlinear in x and y.

We make the linear transformation

$$\begin{pmatrix} z \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

to obtain

$$\bar{f}(z,\bar{z},\mu) = |\lambda(\mu)| \begin{pmatrix} e^{2\pi\phi(\mu)} & 0 \\ 0 & e^{-2\pi\phi(\mu)} \end{pmatrix} \begin{pmatrix} z \\ \bar{z} \end{pmatrix} + \begin{pmatrix} \overline{f_1}(z,\bar{z},\mu) \\ \overline{f_2}(z,\bar{z},\mu) \end{pmatrix},$$

where

$$\overline{f_1}(z,\bar{z},\mu) = f_1(x(z,\bar{z}), y(z,\bar{z}), \mu) + i f_2(x(z,\bar{z}), y(z,\bar{z}), \mu)$$

$$\overline{f_2}(z,\bar{z},\mu) = f_1(x(z,\bar{z}), y(z,\bar{z}), \mu) - i f_2(x(z,\bar{z}), y(z,\bar{z}), \mu).$$

Therefore, all we really need to study is

$$g(z,\bar{z},\mu) = \lambda z + R(z,\bar{z},\mu),$$

where R is nonlinear in z and $\bar{z}$.

Then, if we make the smooth μ-dependent coordinate change

$$w = z + \sum_{j+k=2,3,\ldots,2p+1} \gamma_{jk}(\mu) z^j \bar{z}^k,$$

we will be able to bring g into the form

$$z \to \lambda(\mu) z + \sum_{m=1}^{p} c_{2m+1}(\mu) z^{m+1} \bar{z}^m + O(|z|^{2p+1}).$$

So, in polar coordinates, we have the expression given in the statement of the theorem.

Suppose now that a_{2q+1} is the first of the nonzero coefficients. Then we have

$$g(r,\theta;\mu) = \left((1+\mu)r - a_{2q+1}r^{2m+1} + O(\mu r^{2q+1} + r^{2q+2}),\right.$$
$$\left.\theta + \phi_0 + \phi_1\mu + \sum_{m=1}^{p} b_{2m}r^{2m} + O(\mu^2 + \mu r^2 + r^{2p+1})\right).$$

Observe that the first component of this function,

$$(1+\mu)r - a_{2q+1}r^{2m+1} + O(\mu r^{2q+1} + r^{2q+2}),$$

does not depend on the argument. So, it is a one-parameter family of one-dimensional maps.

One can see, easily, that this family presents a pitchfork bifurcation. If we suppose, for example, that $a_{2q+1} > 0$ (the case $a_{2q+1} < 0$ would be totally analogous), then we can say that there exist two intervals $(\mu_1, 0)$, $(0, \mu_2)$ and $\varepsilon > 0$ such that the following hold:

- If $\mu \in (\mu_1, 0)$, then the corresponding function has a fixed point at the origin that is stable.
- If $\mu \in (0, \mu_2)$, then the corresponding function has three fixed points in the interval $(-\varepsilon, \varepsilon)$. One of them is the origin, and of the two others only one is larger than zero. The origin is unstable and the two others are stable.

Hence, if we translate this situation into our two-dimensional function, it says that the following hold:

- If μ is less than zero, then it does not appear as an invariant circle, since the origin is stable.
- If μ is bigger than zero, then an invariant circle appears, corresponding to a fixed point greater than zero, which the first component has, whose radius is precisely that one-dimensional positive fixed point. Furthermore, that circle is stable like that fixed point. On the other hand, one can see that the origin is, in this case, unstable. ∎

Remark 9. The standard Hopf–Neimark–Sacker bifurcation theorem would correspond to the particular case $p = q = 1$.

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